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ELASTIC MATERIALS OF COAXIAL TYPE AND INEQUALITIES SUFFICIENT TO ENSURE STRONG ELLIPTICITY†

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Abstract—For a class of nonlinearly elastic materials, which is large enough to include all isotropic materials, the condition of strong ellipticity is shown to be equivalent *at certain special states of strain* to strengthened forms of the tension-extension inequalities and the Baker–Ericksen inequalities. We discuss the application of these latter inequalities to the semi-invertibility problem of Truesdell and Moon, to the stability of states of hydrostatic pressure, and to the problem of determining the strain produced by a simple tension.

1. INTRODUCTION

In finite elasticity an *a priori* constitutive inequality is an inequality constraint on the stress (or strain-energy) response function laid down so as to delimit certain classes of physically significant strains and/or to ensure some notion of physically natural material response.‡ Among the oldest and most plausible *a priori* inequalities are the condition of strong ellipticity and, for isotropic materials, the Baker–Ericksen and the tension–extension inequalities. For isotropic elastic materials it has been long known that these three inequalities are not independent: the first, in fact, implies the second and (even a certain strengthened form of) the third.

In the present work, I formulate certain strengthened forms of the Baker–Ericksen and the tension–extension inequalities for a class of materials more general than isotropic, and I show that these new inequalities are still *implied* by the condition of strong ellipticity. More interesting is the fact that these strengthened forms of the Baker–Ericksen and tension–extension inequalities also *imply* the condition of strong ellipticity *at certain special states of strain*, and hence, by a continuity argument, they imply strong ellipticity in an entire neighborhood of these particular strains.

The strengthened Baker–Ericksen inequalities, which, as we say, are implied by the condition of strong ellipicity, turn out to be precisely the ingredient necessary to settle the semi-invertibility problem for stress and strain formulated by Truesdell and Moon[3]. In the section on applications I demonstrate this, and I also remark briefly on particular implications these Baker–Ericksen inequalities have both for the stability of certain states of hydrostatic pressure considered by Varley and Day[4] and for those states of strain, studied by Batra[5] for isotropic materials, which are produced by a simple tension.

2. ELASTIC MATERIALS AND STRONG ELLIPTICITY

Let $\mathscr{B}$ be a body and suppose that the material comprising $\mathscr{B}$ is *elastic* at some particle $X \in \mathscr{B}$. Then, relative to any fixed (reference configuration) κ, there is for X a response function $\mathbf{T}_\kappa(\cdot)$ such that the (symmetric) *Cauchy stress tensor* $\mathbf{T}$ at X is given by

$$\mathbf{T} = \mathbf{T}_\kappa(\mathbf{F}), \tag{1}$$

where $\mathbf{F}$ is the *deformation gradient* at X relative to κ. Denoting by T the set of all tensors (i.e. linear transformations) mapping a three dimensional inner product space V into itself,

†Portions of this work were presented at the 17th Midwestern Mechanics Conference in May 1981.
‡A thorough survey of such inequalities up to 1965 may be found in the treatise[1]. The textbook[2] presents work up to 1973.

we note that the domain D_κ of $\mathbf{T}_\kappa(\cdot)$ is some open† subset of the invertible tensors in T. We will suppose that $\mathbf{T}_\kappa(\cdot)$ is twice continuously differentiable on D_κ.

Let 0^+ denote the set of proper orthogonal tensors in T. We suppose that the response of the material at X is indifferent to superimposed rigid motions, and hence $\mathbf{T}_\kappa(\cdot)$ must meet

$$\mathbf{T}_\kappa(QF) = \mathbf{Q}\mathbf{T}_\kappa(\mathbf{F})\mathbf{Q}^T \quad \forall \mathbf{Q} \in 0^+, \forall \mathbf{F} \in D_\kappa. \tag{2}$$

Note that (2) constitutes a tacit restriction on the domain D_κ, since now we must have $0^+ D_\kappa \subseteq D_\kappa$. Moreover, if we take $\mathbf{Q}$ in (2) to be given by $\mathbf{Q} = \mathbf{Q}(\tau) = \exp(\mathbf{W}\tau)$, $\mathbf{W}$ skew, then differentiation of (2) with respect to τ at $\tau = 0$ gives that at each $\mathbf{F} \in D_\kappa$

$$\partial_F \mathbf{T}_\kappa(\mathbf{F})[\mathbf{W}\mathbf{F}] = \mathbf{W}\mathbf{T}_\kappa(\mathbf{F}) - \mathbf{T}_\kappa(\mathbf{F})\mathbf{W} \tag{3}$$

for every skew tensor $\mathbf{W}$.

Let $\mathbf{a} \otimes \mathbf{b}$ denote the tensor product between any two vectors $\mathbf{a}$ and $\mathbf{b}$ in V. We say that $\mathbf{T}_\kappa(\cdot)$ is **strongly elliptic** at $\mathbf{F} \in D_\kappa$ if

$$\partial_F \mathbf{T}_\kappa(\mathbf{F})[\mathbf{a} \otimes \mathbf{b}\mathbf{F}] \cdot \mathbf{a} \otimes \mathbf{b} > 0 \tag{4}$$

for all unit vectors $\mathbf{a}$ and $\mathbf{b}$ in V. By use of (2) it may be shown that $\mathbf{T}_\kappa(\cdot)$ is strongly elliptic at $\mathbf{F}$ if and only if $\mathbf{T}_\kappa(\cdot)$ is strongly elliptic at $\mathbf{QF}$ for all $\mathbf{Q} \in 0^+$. Moreover, by use of (3) and the symmetry of $\mathbf{T}$, one may easily show that $\mathbf{T}_\kappa(\cdot)$ is strongly elliptic at $\mathbf{F}$ if and only if

$$\frac{1}{2} \partial_F \mathbf{T}_\kappa(\mathbf{F})[(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})\mathbf{F}] \cdot (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}) + \mathbf{T}_\kappa(\mathbf{F}) \cdot (\mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a}) > 0$$

for all unit vectors $\mathbf{a}$ and $\mathbf{b}$. If we interchange $\mathbf{a}$ and $\mathbf{b}$ in this last, we thus see that strong ellipticity at $\mathbf{F}$ for $\mathbf{T}_\kappa(\cdot)$ is the requirement that

$$\frac{1}{2} \partial_F \mathbf{T}_\kappa(\mathbf{F})[(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})\mathbf{F}] \cdot (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}) > |\mathbf{T}_\kappa(\mathbf{F}) \cdot (\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b})| \geqslant 0 \tag{5}$$

for all choices of unit vectors $\mathbf{a}$ and $\mathbf{b}$ in V, i.e.

$$\frac{1}{2} \partial_F \mathbf{T}_\kappa(\mathbf{F})[(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})\mathbf{F}] \cdot (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$$

is always strictly greater than the absolute value of the difference between the normal stress $\mathbf{a} \cdot \mathbf{T}_\kappa(\mathbf{F})\mathbf{a}$ and $\mathbf{b} \cdot \mathbf{T}_\kappa(\mathbf{F})\mathbf{b}$. For brevity, we will often refer to (4) and/or (5) as the "S–E inequality".

While the reader should consult [1, 2] for a thorough study of the physical implications of the S–E inequality, it is useful to recall that, as the name suggests, the S–E inequality is necessary and sufficient for the strong ellipticity at $\mathbf{F}$ of the differential equations of equilibrium for $\mathscr{B}$; additionally, the weakened S–E inequality at $\mathbf{F}$ (i.e. (4) and (5) with "$>$" replaced by "$\geqslant$") is a necessary condition for the Hadamard infinitesimal stability of any deformation field having $\mathbf{F}$ as one of its values. Further, as Gurtin and Spector[6] have shown, if $\boldsymbol{\kappa}$ is a homogeneous reference configuration for a homogeneous elastic body, then the S–E inequality at $\mathbf{1}$ for $\mathbf{T}_\kappa(\cdot)$ is sufficient for the existence of a neighborhood of deformations about the identity map, $\imath(\cdot)$: $\boldsymbol{\kappa} \to \boldsymbol{\kappa}$, all of which are uniformly Hadamard stable‡ with respect to processes which leave the boundary of $\boldsymbol{\kappa}$ fixed. Finally, for elastic

†With tr(·) denoting the usual trace operator on T, we make T an inner product space by setting $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr} \mathbf{A}\mathbf{B}^T$ for $\mathbf{A}$ and $\mathbf{B}$ in T.

‡See [6] for precise definitions.

materials that possess a strain energy, the S–E inequality is necessary and sufficient for the squared wave speeds to be positive for every direction of propagation.†

Of the various *a priori* inequalities in finite elasticity, the S–E inequality is one of the more commonly used to delimit physically reasonable deformation states F and/or response functions $\mathbf{T}_\kappa(\cdot)$. As a *necessary* condition on the physically reasonable (e.g. statically realizable) deformations in solid materials, the S–E (or weakened S–E) inequality certainly seems appropriate; that it alone, however, is not *sufficient* to completely delimit the class of such deformations is suggested by the analysis in [1, 2]. Additionally, the note[9] demonstrates that rather queer, unrealistic, elastic materials can satisfy the S–E inequality (as well as certain other *a priori* inequalities) over a large class of deformation gradients F. It is, however, the plausible necessity of strong ellipticity for realistic behavior, rather than its unfortunate insufficiency, that engages us here. Specifically, we seek necessary and sufficient conditions for the S–E inequality to hold at certain states of strain in a broad class of elastic materials.‡

3. ELASTIC MATERIALS OF COAXIAL TYPE

Let $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^T$ be the (positive definite and symmetric) left Cauchy–Green strain tensor for the deformation gradient F. The elastic material at $X \in \mathscr{B}$ will be said to be of **coaxial type** if, for some reference configuration κ, the stress $\mathbf{T}_\kappa(\mathbf{F})$ commutes with B at each $\mathbf{F} \in D_\kappa$, i.e.

$$\mathbf{T}_\kappa(\mathbf{F})\mathbf{B} = \mathbf{B}\mathbf{T}_\kappa(\mathbf{F}) \;\; \forall \;\; \mathbf{F} \in D_\kappa. \tag{6}$$

We say then that the strain $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and the stress $\mathbf{T} = \mathbf{T}_\kappa(\mathbf{F})$ are *coaxial*, and, as is well-known, the coaxiality of B and T is equivalent to their sharing a common, orthonormal basis of eigenvectors, say $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $i = 1, 2, 3$.

We note that the condition (6) is consistent with the material's indifference to superimposed rigid motions as embodied in (2) in the sense that, if (6) holds for even a single $\mathbf{F} \in D_\kappa$, it will then by (2) hold automatically for all QF, $\mathbf{Q} \in 0^+$. We also note that (6) depends in an essential way on the underlying reference configuration κ—if (6) holds for a given reference configuration κ, it will generally fail to hold for an arbitrary second reference configuration $\bar{\kappa}$. However, since $\mathbf{T}_{\bar{\kappa}}(\cdot) = \mathbf{T}_\kappa((\cdot)\mathbf{G})$ where G is the gradient at X of the deformation from κ to $\bar{\kappa}$, it is easy to see that if (6) holds for κ then (6) will also hold for $\bar{\kappa}$ provided $\mathbf{G} = \alpha\mathbf{R}$, R orthogonal and $\alpha \neq 0$. In particular, the property (6) is unchanged by a dilatation $\mathbf{G} = \alpha\mathbf{1}$ of the reference configuration. By analogy with isotropic materials (see below), we will call a configuration κ such that (6) holds **undistorted** (at the particle X).

Isotropic elastic materials provide a major special case of materials of coaxial type since for an isotropic material there are reference configurations κ (called undistorted) such that

$$\mathbf{T} = \mathbf{T}_\kappa(\mathbf{F}) = N_0(\mathbf{B})\mathbf{1} + N_1(\mathbf{B})\mathbf{B} + N_2(\mathbf{B})\mathbf{B}^2,$$

where the functions $N_i(\cdot)$, defined on a subset of the positive definite, symmetric tensors, are isotropic. While it is easily verified that isotropic materials satisfy (6), we note that they are but a special case of those materials of coaxial type for which

$$\mathbf{T} = \mathbf{T}_\kappa(\mathbf{F}) = M_0(\mathbf{F})\mathbf{1} + M_1(\mathbf{F})\mathbf{B} + M_2(\mathbf{F})\mathbf{B}^2, \tag{7}$$

where, to satisfy (2), we suppose the functions $M_i(\cdot)$ to satisfy $M_i(\mathbf{Q}\mathbf{F}) = M_i(\mathbf{F})$ for all $\mathbf{Q} \in 0^+$

†In addition to [1, 2], mention should be made here of the work of Sawyers and Rivlin[7] on wave propagation. See also the review article[8] by Rivlin.

‡Note added. After our manuscript was submitted for publication, Professor S. Spector showed us his manuscript[10] with Simpson. There the much more difficult task of finding necessary and sufficient conditions for the S–E inequality to hold at an *arbitrary* deformation F in an isotropic elastic material is solved in the sense that (4) is shown to be equivalent to at most 12 independent scalar inequalities involving only the components of $\partial_F\mathbf{T}_\kappa(\mathbf{F})$. The physical interpretation of 6 of these inequalities remains, however, a difficult open problem. Here, by considering a much smaller class of strains, we are able to obtain, for a somewhat larger class of materials, much simpler necessary and sufficient conditions for the S–E inequality to hold.

and $F \in D_\kappa$. It will occasionally be useful to apply some of our general results to the form (7) and, *a fortiori*, to isotropic materials.

4. THE B–E$^+$ and T–E$^+$ INEQUALITIES

The B–E (for Baker–Ericksen) inequalities are well-known and have been much studied† in the context of isotropic materials. We now formulate these inequalities for materials of coaxial type as well as extend and sharpen them for states of strain **B** that possess double or triple eigenvalues. Thus, for a given deformation gradient **F**, let $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $i = 1, 2, 3$, be common orthonormal eigenvectors shared by **T(F)**‡ and **B**, and let t_i and β_i be, respectively, their corresponding associated eigenvalues. We will say that the **B–E inequalities** hold at **F** for $\mathbf{T}(\cdot)$ if the greater principal tension at **F** occurs in the direction of the greater principal extension at **F**, i.e. if for each i and j, $i \neq j$,

$$\beta_i > \beta_j \Rightarrow t_i > t_j, \tag{8$_1$}$$

or, equivalently, if

$$\frac{t_i - t_j}{\beta_i - \beta_j} > 0 \text{ for } \beta_i \neq \beta_j. \tag{8$_2$}$$

In terms of those materials of coaxial type given by (7), we see then that the B–E inequalities hold at **F** if and only if

$$M_1(\mathbf{F}) + (\beta_i + \beta_j)M_2(\mathbf{F}) > 0$$

for each β_i and β_j with $\beta_i \neq \beta_j$.

Now, if some t_i equals some t_j at a deformation **F** where the B–E inequalities hold, then it is clear from $(8)_1$ that β_i must equal β_j; however, as they stand the B–E inequalities assert nothing in the case that some β_i and some β_j coincide, as will occur if **B** has double or triple eigenvalues. We will show below, however, that for materials of coaxial type $t_i = t_j$ whenever $\beta_i = \beta_j$, and then, while the condition $(8)_1$ suggests little, the form $(8)_2$ is suggestive of a limit condition which we now formulate. Indeed, given a sequence $\{\mathbf{S}_n\}$ of symmetric tensors with limit **S**, we say that $\{\mathbf{S}_n\}$ is *tame* if for each $\mathbf{S}_n$ there exists orthonormal eigenvectors $\mathbf{e}_i(n)$ of $\mathbf{S}_n$, $i = 1, 2, 3$, such that each of the three sequences $\{\mathbf{e}_i(n)\}$ has a limit, say $\mathbf{e}_i$, as $n \to \infty$. Such eigenvectors $\mathbf{e}_i(n)$ of $\mathbf{S}_n$ will be called *regular*, and it is easy to verify that the limit $\mathbf{e}_i$ of $\{\mathbf{e}_i(n)\}$ is an eigenvector of $\mathbf{S} = \lim_{n\to\infty} \mathbf{S}_n$, with a corresponding eigenvalue of $\lim_{n\to\infty} \mathbf{e}_i(n) \cdot \mathbf{S}_n \mathbf{e}_i(n)$. We will say that a sequence of deformation gradients $\{\mathbf{F}_n\}$ with limit **F** is *tame* and *proper* if the sequence $\{\mathbf{B}_n\}$, $\mathbf{B}_n \equiv \mathbf{F}_n\mathbf{F}_n^T$, with limit $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is tame and if each $\mathbf{B}_n$ has distinct eigenvalues.§ For materials of coaxial type it is clear that any tame and proper sequence $\{\mathbf{F}_n\}$ gives rise to three regular, orthonormal eigenvectors $\mathbf{e}_i(n)$ of $\mathbf{B}_n$ that are also (regular) eigenvectors of $\mathbf{T}(\mathbf{F}_n)$. If we let $t_i(n)$ and $\beta_i(n)$ denote the eigenvalues of, respectively, $\mathbf{T}(\mathbf{F}_n)$ and $\mathbf{B}_n$ associated with such $\mathbf{e}_i(n)$, we see that it then makes sense to say that the **B–E$^+$** (for strengthened Baker–Ericksen) **inequalities** hold for $\mathbf{T}(\cdot)$ at a deformation **F** if, for every tame and proper sequence $\{\mathbf{F}_n\}$ tending to **F**, we have that

$$\lim_{n\to\infty} \frac{t_i(n) - t_j(n)}{\beta_i(n) - \beta_j(n)} \quad \text{exists and is positive} \tag{9}$$

for each i and j, $i \neq j$. Since $\{t_i(n)\}$ and $\{\beta_i(n)\}$ each have a limit which is an eigenvalue of, respectively, **T(F)** and **B**, it is clear that (9) and (8) are equivalent when

†See [1, 2].
‡Note that, until further notice, we suppress the dependence of $\mathbf{T}_\kappa(\cdot)$ on the (undistorted) reference κ.
§Note that the limit **B** of $\{\mathbf{B}_n\}$ need *not* have distinct eigenvalues.

$\lim_{n\to\infty} \beta_i(n) \neq \lim_{n\to\infty} \beta_j(n)$, and that (9) is consistent with (8) when $\lim_{n\to\infty} t_i(n) = \lim_{n\to\infty} t_j(n)$. However, (9) is in general a more stringent condition than (8) since it is meaningful even when $\beta_i(n) - \beta_j(n) \to 0$ as $n \to \infty$. Indeed, for materials of coaxial type that meet (7), the B–E$^+$ inequalities hold at a deformation **F** if and only if for each i and j, $i \neq j$,

$$M_1(\mathbf{F}) + (\beta_i + \beta_j)M_2(\mathbf{F}) > 0$$

—regardless of whether β_i equals β_j or not.†

Proposition 1. *For a material of coaxial type, let* $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $i = 1$ or 2, *be any pair of orthonormal eigenvectors shared by* $\mathbf{T}(\mathbf{F})$ *and* $\mathbf{B}$, *and let* $t_i = t_i(\mathbf{F})$ *and* $\beta_i = \beta_i(\mathbf{F})$ *be their corresponding respective eigenvalues. Then,*

$$\beta_i = \beta_j \Rightarrow t_i = t_j,$$

and, if the B–E *or* B–E$^+$ *inequalities hold at* **F**,

$$t_i = t_j \Rightarrow \beta_i = \beta_j.$$

Moreover, the B–E$^+$ *inequalities hold at* **F** *if and only if*

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}\mathbf{F}]\cdot\mathbf{a}\otimes\mathbf{b} > 0 \tag{10}$$

for every pair of orthonormal eigenvectors, **a** *and* **b**, *shared by* $\mathbf{T}(\mathbf{F})$ *and* **B**.

Upon comparing (4) and (10), we see that Proposition 1 has the following

Corollary. *If a material of coaxial type satisfies the* S–E *inequality at* **F**, *it also satisfies the* B–E$^+$ *inequalities at* **F**.‡

Proof: Let $\mathbf{F} = \mathbf{F}(\tau)$ be a smooth path of deformations and differentiate $\mathbf{T}(\mathbf{F})\mathbf{B} = \mathbf{B}\mathbf{T}(\mathbf{F})$ with respect to τ. We thus find that

$$\dot{\mathbf{T}}\mathbf{B} + \mathbf{T}\dot{\mathbf{B}} = \mathbf{B}\dot{\mathbf{T}} + \dot{\mathbf{B}}\mathbf{T}, \tag{11}$$

where $\dot{\mathbf{T}} = \partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\dot{\mathbf{F}}]$, $\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T$, **F** is any tensor in the domain D of $\mathbf{T}(\cdot)$, and where $\dot{\mathbf{F}}$ is arbitrary since D is open. Upon taking $\dot{\mathbf{F}} = \mathbf{e}_i\otimes\mathbf{e}_j\mathbf{F}$, we see that $\dot{\mathbf{B}} = \beta_j(\mathbf{e}_i\otimes\mathbf{e}_j + \mathbf{e}_j\otimes\mathbf{e}_i)$, and therefore the inner produce of (11) with $\mathbf{e}_i\otimes\mathbf{e}_j$ gives that

$$\dot{\mathbf{T}}\cdot\mathbf{e}_i\otimes\mathbf{e}_j\{\beta_i - \beta_j\} = \beta_j\{t_i - t_j\} \tag{12}$$

where $\dot{\mathbf{T}} = \partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{e}_i\otimes\mathbf{e}_j\mathbf{F}]$. It is thus clear that $t_i = t_j$ whenever $\beta_i = \beta_j$.

Now let $\{\mathbf{F}_n\}$ be a tame and proper sequence with limit **F** and apply (12) at each deformation $\mathbf{F}_n$. We find that

$$\frac{t_i(n) - t_j(n)}{\beta_i(n) - \beta_j(n)} = \frac{1}{\beta_j(n)}\partial_{\mathbf{F}}T(\mathbf{F}_n)[\hat{\mathbf{e}}_i(n)\otimes\hat{\mathbf{e}}_j(n)\mathbf{F}_n]\cdot\hat{\mathbf{e}}_i(n)\otimes\hat{\mathbf{e}}_j(n),$$

where $t_i(n)$ and $\beta_i(n)$ are the eigenvalues of, respectively, $\mathbf{T}(\mathbf{F}_n)$ and $\mathbf{B}_n$ associated with their shared regular eigenvector $\hat{\mathbf{e}}_i(n)$, $i = 1, 2, 3$. We see therefore that

$$\operatorname*{Lim}_{n\to\infty}\frac{t_i(n) - t_j(n)}{\beta_i(n) - \beta_j(n)} = \frac{1}{\beta_j}\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\mathbf{F}]\cdot\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j, \tag{13}$$

†Here, of course, β_i and β_j are those eigenvalues of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ given by the limit as $n \to \infty$ of the sequences $\{\beta_i(n)\}$ and $\{\beta_j(n)\}$, respectively.

‡That S–E ⇒ B–E for isotropic materials is well-known (see [1, 2]).

where $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ are the limits of $\{\hat{\mathbf{e}}_i(n)\}$ and $\{\hat{\mathbf{e}}_j(n)\}$, respectively, and are shared orthonormal eigenvectors of $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$, and where β_j is the eigenvalue of $\mathbf{B}$ corresponding to $\hat{\mathbf{e}}_j$. That (10) is sufficient for the B–E$^+$ inequalities at $\mathbf{F}$ is now clear.

To see that (10) is also necessary for the B–E$^+$ inequalities at $\mathbf{F}$, let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be orthonormal eigenvectors of $\mathbf{B}$ with β_a, β_b and β_c as their respective associated eigenvalues. The sequence $\{\mathbf{F}_n\}$ given by

$$\mathbf{F}_n = \{\sqrt{\beta_a + 1/n}\,\mathbf{a}\otimes\mathbf{a} + \sqrt{\beta_b + 1/n^2}\,\mathbf{b}\otimes\mathbf{b} + \sqrt{\beta_c - 1/n^3}\,\mathbf{c}\otimes\mathbf{c}\}\mathbf{B}^{-1/2}\mathbf{F},$$

has a limit of $\mathbf{F}$ and, since

$$\mathbf{B}_n = \mathbf{F}_n\mathbf{F}_n^T = \{(\beta_a + 1/n)\mathbf{a}\otimes\mathbf{a} + (\beta_b + 1/n^2)\mathbf{b}\otimes\mathbf{b} + (\beta_c + 1/n^3)\mathbf{c}\otimes\mathbf{c}\},$$

$\{\mathbf{F}_n\}$ is also tame and proper for n large enough. If we now apply (13) to this sequence $\{\mathbf{F}_n\}$ we easily see that (10) must hold whenever the B–E$^+$ inequalities hold at $\mathbf{F}$.

Finally, we see by (10) and (12) that, if the B–E$^+$ inequalities hold at $\mathbf{F}$, then $t_i = t_j$ *only if* $\beta_i = \beta_j$.■

The T–E$^+$ (for strengthened tension-extension) inequalities are also well-known in the context of isotropic elastic materials.† To formulate these inequalities for materials of coaxial type, consider a deformation state $\mathbf{F}$ and let $\mathbf{e} = \mathbf{e}(\mathbf{F})$ be one of the unit eigenvectors shared by $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$. Let the material in the state $\mathbf{F}$ be subjected to a further deformation consisting of a simple extension along the direction $\mathbf{e}$ of amount $\alpha(> -1)$ so that its deformation state is now $\mathbf{F}(\alpha) = (\mathbf{1} + \alpha\mathbf{e}\otimes\mathbf{e})\mathbf{F}$. Since $\mathbf{B}(\alpha) = \mathbf{F}(\alpha)\mathbf{F}(\alpha)^T = \mathbf{B} + \beta(\alpha^2 + 2\alpha)\mathbf{e}\otimes\mathbf{e}$, where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and β is that eigenvalue of $\mathbf{B}$ corresponding to $\mathbf{e}$, we see that $\mathbf{e}$ will also be an eigenvector of $\mathbf{B}(\alpha)$ with $\beta(\alpha + 1)^2$ as its corresponding eigenvalue. Moreover, the remaining eigenvalues and eigenvectors of $\mathbf{B}(\alpha)$ will be those remaining to $\mathbf{B}$. It follows therefore that, for $\alpha \neq 0$ and small enough, the characteristic space of $\mathbf{B}(\alpha)$ corresponding to $\beta(\alpha + 1)^2$ will just be the line generated by $\mathbf{e}$.‡ But $\mathbf{B}(\alpha)\{\mathbf{T}(\mathbf{F}(\alpha))\mathbf{e}\} = \beta(\alpha + 1)^2\{\mathbf{T}(\mathbf{F}(\alpha))\mathbf{e}\}$ since $\mathbf{T}(\mathbf{F}(\alpha))$ and $\mathbf{B}(\alpha)$ commute, and therefore for α small enough we conclude that

$$\mathbf{T}(\mathbf{F}(\alpha))\mathbf{e} = t(\alpha)\mathbf{e},‡$$

where $t(0) = t$ is the eigenvalue of $\mathbf{T}(\mathbf{F})$ corresponding to the eigenvector $\mathbf{e}$ shared between $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$. Thus, $\mathbf{e}$ is not only an eigenvector of $\mathbf{B}(\alpha)$—it is also an eigenvector of $\mathbf{T}(\mathbf{F}(\alpha))$ whenever α is non-zero and small enough, and it is plausible that the associated tension $t(\alpha) = \mathbf{e}\cdot\mathbf{T}(\mathbf{F}(\alpha))\mathbf{e}$ increase or decrease as α, the amount of extension, is increased or decreased. When this holds in the strict sense that$(d/d\alpha)t(\alpha)|_{\alpha=0} > 0$ for extensions in *each* of the principal directions shared by $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$, we will say that the **T–E$^+$ inequalities** hold for $\mathbf{T}(\cdot)$ at $\mathbf{F}$. Hence, for materials of coaxial type, the T–E$^+$ inequalities for $\mathbf{T}(\cdot)$ hold at a deformation $\mathbf{F}$ if and only if

$$\frac{d}{d\alpha}\,\mathbf{e}\cdot\mathbf{T}((\mathbf{1} + \alpha\mathbf{e}\otimes\mathbf{e})\mathbf{F})\mathbf{e}\bigg|_{\alpha=0} = \mathbf{e}\cdot\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{e}\otimes\mathbf{e}\mathbf{F}]\mathbf{e} > 0 \qquad (14)$$

for each of the principal directions $\mathbf{e}$ shared by $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$. The form taken by (14) for the materials of coaxial type given by (7) is easy but unenlightening to write down.

Upon comparing (14) to (4) with $\mathbf{a}\otimes\mathbf{b} = \mathbf{e}\otimes\mathbf{e}$, we arrive at once at

Proposition 2. *If a material of coaxial type satisfies the* S–E$^+$ *inequality at* $\mathbf{F}$, *it also satisfies the* T–E$^+$ *inequalities at* $\mathbf{F}$.§

†Again, see[1, 2].

‡Indeed, this will be true for all values of α (small or not) other than those for which $\beta(\alpha + 1)^2$ is an eigenvalue of $\mathbf{B}$.

§That S–E $\Rightarrow$ T–E$^+$ for isotropic materials is well known (see [1, 2]).

5. WHEN DOES B-E$^+$ AND T-E$^+ \Rightarrow$S-E?

Consider now a material of coaxial type which is in a spherical state of strain $\mathbf{B} = \beta\mathbf{1}$, $\beta > 0$, relative to an undisorted reference configuration. By Proposition 1 the stress $\mathbf{T}(\mathbf{F})$ will also now be spherical, where the associated deformation gradient $\mathbf{F}$ is of the form $\sqrt{\beta}\mathbf{R}$,$\mathbf{R}$ orthongonal. For a large class materials of coaxial type, including all those which are isotropic, we now show that at such a state of strain Proposition 2 and the Corollary to Proposition 1 have as converse that satisfaction of the B–E$^+$ and T–E$^+$ inequalities at $\mathbf{F}$ implies that the S–E inequality holds at $\mathbf{F}$. Thus, by an easy continuity argument, the B–E$^+$ and T–E$^+$ inequalities at deformation $\mathbf{F} = \sqrt{\beta}\mathbf{R}$, $\mathbf{R}$ orthogonal, ensures the strong ellipticity of $\mathbf{T}(\cdot)$ in an entire neighborhood of $\mathbf{F}$.

Our demonstration requires that we have a representation of $\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[(\cdot)\mathbf{F}]$ at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ for any material of coaxial type. Since $\mathbf{T}(\mathbf{F})$ is spherical for such $\mathbf{F}$, we see by (3) that

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{W}\mathbf{F}] \equiv 0$$

for all skew tensors $\mathbf{W}$. It only remains therefore to find the form taken by $\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[(\cdot)\mathbf{F}]$, $\mathbf{F} = \sqrt{\beta}\mathbf{R}$, on the set T_S of symmetric tensors. To achieve this, let us differentiate (11) with respect to τ to find that

$$\ddot{\mathbf{T}}\mathbf{B} + 2\dot{\mathbf{T}}\dot{\mathbf{B}} + \mathbf{T}\ddot{\mathbf{B}} = \mathbf{B}\ddot{\mathbf{T}} + 2\dot{\mathbf{B}}\dot{\mathbf{T}} + \ddot{\mathbf{B}}\mathbf{T}$$

for any smooth path $\mathbf{F}(\tau)$. If we evaluate this at $\tau = 0$ on a path $\mathbf{F}(\tau)$ for which $\mathbf{F}(0) = \sqrt{\beta}\mathbf{R}$, $\mathbf{R}$ orthogonal, we find that

$$\dot{\mathbf{T}}\dot{\mathbf{B}} = \dot{\mathbf{B}}\dot{\mathbf{T}},$$

since $\mathbf{B}(0)$ and $\mathbf{T}(\mathbf{F}(0))$ are both spherical, and where $\dot{\mathbf{T}} = \partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\dot{\mathbf{F}}]$, $\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T$, $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ and where $\dot{\mathbf{F}}$ is arbitrary. If we now take $\dot{\mathbf{F}} = \mathbf{S}\mathbf{F}$, $\mathbf{S}$ symmetric, we see that $\dot{\mathbf{B}} = 2\beta\mathbf{S}$ and, hence, that

$$\mathscr{L}(\mathbf{S})\mathbf{S} = \mathbf{S}\mathscr{L}(\mathbf{S}) \ \forall \ \mathbf{S} \in T_S, \tag{15}$$

where $\mathscr{L}(\cdot)$ is the linear map on T_S into T_S given by $\mathscr{L}(\cdot) \equiv \partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[(\cdot)\mathbf{F}]$, $\mathbf{F} = \sqrt{\beta}\mathbf{R}$. It can be shown that (15) implies that the map $\mathscr{L}(\cdot)$ is of the form $\mathscr{L}(\mathbf{S}) = 2\mu\mathbf{S} + (\Lambda \cdot \mathbf{S})\mathbf{1}$ for some number μ and some symmetric tensor Λ. Thus, in any material of coaxial type,

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{S}\mathbf{F}] = 2\mu\mathbf{S} + (\Lambda \cdot \mathbf{S})\mathbf{1} \ \forall \ \mathbf{S} \in T_S, \tag{16}$$

whenever $\mathbf{F} = \sqrt{\beta}\mathbf{R}$, $\mathbf{R}$ orthogonal, and where $\mu = \mu(\mathbf{F})$ and $\Lambda = \Lambda(\mathbf{F}) = \Lambda(\mathbf{F})^T$. We remark that if the material is also isotropic then it can also be shown that Λ must be spherical, $\Lambda = \lambda\mathbf{1}$.

Now at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ every direction is a shared principal direction of $\mathbf{T}(\mathbf{F})$ and $\mathbf{B}$. Hence, by (16) and Proposition 1, the B–E$^+$ inequalities will hold at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ if and only if

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}\mathbf{F}]\cdot\mathbf{a}\otimes\mathbf{b} = \mu > 0 \tag{17}$$

for every pair of orthonormal vectors, $\mathbf{a}$ and $\mathbf{b}$. Similarly, from (16) and (14), we see that the T–E$^+$ inequalities will hold at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ if and only if

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[\mathbf{e}\otimes\mathbf{e}\mathbf{F}]\cdot\mathbf{e}\otimes\mathbf{e} = 2\mu + \Lambda\cdot\mathbf{e}\otimes\mathbf{e} > 0 \tag{18}$$

for every unit vector $\mathbf{e}$, and it is easy to show that this equivalent to the requirement that $2\mu + \lambda > 0$ for each eigenvalue λ of Λ. Lastly, by (16) and (5), it is clear that the S–E inequality will hold at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ if and only if

$$\partial_{\mathbf{F}}\mathbf{T}(\mathbf{F})[(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})\mathbf{F}]\cdot(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}) = 4\{\mu\{1 + (\mathbf{a}\cdot\mathbf{b})^2\} + (\mathbf{a}\cdot\Lambda\mathbf{b})(\mathbf{a}\cdot\mathbf{b})\} > 0 \tag{19}$$

for all unit vectors $\mathbf{a}$ and $\mathbf{b}$.

As we already know (and as is easily verified directly), (17) and (18) are implied by (19). Under what conditions do (17) and (18) imply (19), i.e. what are necessary and sufficient conditions on μ and Λ for (19) to hold? To formulate such conditions, we note that, in terms of the symmetric tensor

$$\mathbf{P} = \mathbf{P}(\mathbf{b}) \equiv 2\mu(\mathbf{1} + \mathbf{b}\otimes\mathbf{b}) + \Lambda\mathbf{b}\otimes\mathbf{b} + \mathbf{b}\otimes\Lambda\mathbf{b},$$

(19) is the requirement that for each unit vector $\mathbf{b}$

$$\mathbf{a}\cdot\mathbf{P}\mathbf{a} > 0 \;\; \forall \text{ unit vectors } \mathbf{a}.$$

That is, $\mathbf{P}(\mathbf{b})$ must be a positive definite tensor for every unit vector $\mathbf{b}$, and we see then that (19) is the requirement that the eigenvalues of $\mathbf{P}(\mathbf{b})$ be positive for every unit vector $\mathbf{b}$.

Now any vector perpendicular to $\mathbf{b}$ and $\Lambda\mathbf{b}$ is easily seen to be an eigenvector of $\mathbf{P}(\mathbf{b})$ with an associated eigenvalue of 2μ. We thus rediscover the necessity of the condition (17) if (19) is to hold. Additionally, we see that if $\Lambda\mathbf{b}$ is parallel to $\mathbf{b}$, $\Lambda\mathbf{b} = \lambda\mathbf{b}$ for some (eigenvalue of Λ) λ, then 2μ is a double eigenvalue of $\mathbf{P}(\mathbf{b})$, and $2\mu + \lambda = 2\mu + \Lambda\cdot\mathbf{b}\otimes\mathbf{b}$ is the remaining eigenvalue of $\mathbf{P}(\mathbf{b})$. We have thus also rediscovered the necessity of (18) for (19). More interesting is the case when $\mathbf{b}$ is not an eigenvector of Λ so that $\mathbf{b}$ and $\Lambda\mathbf{b}$ are linearly independent. In this case, beyond the unit eigenvector corresponding to 2μ, there are two orthonormal eigenvectors of $\mathbf{P}(\mathbf{b})$ in the plane of $\mathbf{b}$ and $\Lambda\mathbf{b}$. The characeristic equation for the two corresponding eigenvalues, p_1 and p_2, is easily claculated to be

$$p^2 - (6\mu + 2\mathbf{b}\cdot\Lambda\mathbf{b})p + 8\mu^2 + 4\mu\mathbf{b}\cdot\Lambda\mathbf{b} + (\mathbf{b}\cdot\Lambda\mathbf{b})^2 - \mathbf{b}\cdot\Lambda^2\mathbf{b} = 0,$$

and, since its roots p_1 and p_2 will be positive if and only if $p_1 + p_2$ and $p_1 p_2$ are both positive, we see that

$$3\mu + \mathbf{b}\cdot\Lambda\mathbf{b} > 0, \tag{20$_1$}$$

and

$$8\mu^2 + 4\mu\mathbf{b}\cdot\Lambda\mathbf{b} + (\mathbf{b}\cdot\Lambda\mathbf{b})^2 - \mathbf{b}\cdot\Lambda^2\mathbf{b} > 0, \tag{20$_2$}$$

along with (17) and (18), are the necessary and sufficient conditions that $\mathbf{P}(\mathbf{b})$ be positive definite for every unit vector $\mathbf{b}$.

Now the condition $(20)_1$ is easily seen to be implied by (17) and (18), and thus it need be no longer considered. The condition $(20)_2$, while derived under the assumption that $\mathbf{b}$ was not an eigenvector of Λ, is also easily seen to be implied by (17) and (18) in the special case when $\mathbf{b}$ is an eigenvector of Λ. Thus (17), (18), and the requirement that

$$\phi(\mathbf{b}) \equiv 8\mu^2 + 4\mu\mathbf{b}\cdot\Lambda\mathbf{b} + (\mathbf{b}\cdot\Lambda\mathbf{b})^2 - \mathbf{b}\cdot\Lambda^2\mathbf{b} > 0 \tag{21}$$

for all unit vectors $\mathbf{b}$, are together necessary and sufficient conditions for $\mathbf{P}(\mathbf{b})$ to be positive definite for all unit $\mathbf{b}$, and hence they are necessary and sufficient for the S–E inequality to hold at $\mathbf{F} = \sqrt{\beta}\mathbf{R}$, $\mathbf{R}$ orthongonal. Moreover, since (17) and (18) imply that $\phi(\mathbf{b})$ is positive whenever $\mathbf{b}$ is an eigenvector of Λ, to ensure (21) we need only examine the local minimums of $\phi(\cdot)$ taken on at unit vectors $\mathbf{b}$ that are not eigenvectors of Λ. *A fortiori*, we may dispense with (21) altogether if Λ is spherical†—at any deformation $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ for which Λ is spherical, (17) and (18) are by themselves necessary and sufficient for (19), i.e. S–E *at* $\mathbf{F}$ *if and only if* B–E$^+$ *and* T–E$^+$ at $\mathbf{F}$.

In the more complex case when Λ has 2 or 3 distinct eigenvalues, an analysis of the

†As it is in every isotropic material.

local extrema of $\phi(\cdot)$ via the method of Lagrange multipliers yields the following result: Additional extreme values of $\phi(\cdot)$, beyond those (positive) extrema taken on at the eigenvectors of A, occur if and only if some pair, λ and $\bar{\lambda}$, of unequal eigenvalues of A meets

$$|\lambda - \bar{\lambda}| > 4\mu,$$

and in this case the corresponding extreme value of $\phi(\cdot)$ is

$$4\mu^2 + 2\mu(\lambda + \bar{\lambda}) - \frac{1}{4}(\lambda - \bar{\lambda})^2.$$

Hence, given (17) and (18), the condition (21) will hold if and only if every pair, λ and $\bar{\lambda}$, of eigenvalues of A meets

$$|\lambda - \bar{\lambda}| \leq 4\mu \quad \text{or} \quad 4\mu^2 + 2\mu(\lambda + \bar{\lambda}) - \frac{1}{4}(\lambda - \bar{\lambda}^2) > 0. \tag{22}$$

We summarize the above discussion in

Proposition 3. *In any material of coaxial type and at any deformation* $\mathbf{F} = \sqrt{\beta}\mathbf{R}$ *relative to an undistorted reference state,* $\mathbf{R}$ *orthogonal, the* S–E *inequality holds if and only if the* B–E$^+$ *inequalities, the* T–E$^+$ *inequalities, and* (22) *hold, where* μ *and* A *are the material parameters appearing in the representation* (16)

If we view (22) as a material restriction, then we have the following

Corollary. *For all materials of coaxial type for which* (22) *holds at* $\mathbf{F} = \sqrt{\beta}\mathbf{R}$, *the* S–E *inequality holds at* $\mathbf{F}$ *if and only if the* B–E$^+$ *and* T–E$^+$ *inequalities hold at* $\mathbf{F}$. *Isotropic materials are included as a special case.*

6. APPLICATIONS

Here I apply the ideas of the last two sections to some problems suggested by recent work of Truesdell and Moon[3], Varley and Day[4], and of Batra[5].

Let the material at some particle $X \in \mathscr{B}$ be elastic and of coaxial type, and let κ be one of its undistorted configurations. Let $\hat{\kappa}$ be a second configuration of $\mathscr{B}$ such that in $\hat{\kappa}$ the stress system at X is spherical, $\mathbf{T} = \hat{t}\mathbf{1}$, and let $\hat{\mathbf{F}}$ be the gradient at X of the deformation from κ to $\hat{\kappa}$. *If* $\mathbf{T}_\kappa(\cdot)$ *satisfies either the* B–E *or* B–E$^+$ *inequalities at* $\hat{\mathbf{F}}$, *then* $\hat{\kappa}$ *is also an undistorted configuration at* X.† Indeed, to see this we need only note that, since $\mathbf{T}_\kappa(\hat{\mathbf{F}}) = \hat{t}\mathbf{1}$ is spherical, Proposition 1 tells us that it is necessary for $\hat{\mathbf{B}} \equiv \hat{\mathbf{F}}\hat{\mathbf{F}}^T$ to also be spherical, $\hat{\mathbf{B}} = \hat{\beta}\mathbf{1}$, since $\mathbf{T}_\kappa(\cdot)$ satisfies either the B–E or B–E$^+$ inequalities at $\hat{\mathbf{F}}$. Hence, $\hat{\mathbf{F}} = \sqrt{\hat{\beta}}\mathbf{R}$ for some orthogonal tensor $\mathbf{R}$, and, since $\mathbf{T}_{\hat{\kappa}}(\cdot) = \mathbf{T}_\kappa((\cdot)\hat{\mathbf{F}})$, it is now clear that (6) will hold for $\hat{\kappa}$ as well as κ.

The above simple remarks also show that a spherical state of stress at a deformation $\mathbf{F}$ for which the strain $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is *not* spherical is possible only if the B–E and B–E$^+$ inequalities *fail* at $\mathbf{F}$. *A fortiori*, the strong ellipticity of $\mathbf{T}_\kappa(\cdot)$ must also fail at such an $\mathbf{F}$. To the extent then the strong ellipticity is a "stability" criterion, we may thus assert that non-spherical states of strain $\mathbf{B}$ at which the stress is spherical are always "unstable" in any material of coaxial type. The existence, if not the stability, of such states was considered by Varley and Day in[4].

In[3] Truesdell and Moon studied conditions under which an isotropic material, in particular, would have a "semi-invertible" stress–strain relation. It is a simple generalization of their idea to say that a material of coaxial type of the form given in (7) has a **semi-invertible** response function $\mathbf{T}_\kappa(\cdot)$ at $\mathbf{F}$ if there exists $H_i = H_i(\mathbf{F})$, $i = 1, 2, 3$, such that

$$\mathbf{B} = H_0(\mathbf{F})\mathbf{1} + H_1(\mathbf{F})\mathbf{T} + H_2(\mathbf{F})\mathbf{T}^2, \tag{23}$$

where $\mathbf{T}$, $\mathbf{F}$, and $\mathbf{B}(= \mathbf{F}\mathbf{F}^T)$ satisfy (7). As Truesdell and Moon remarked, even in isotropic

†See Truesdell and Moon ([3], p. 189).

materials the B–E inequalities do not in general suffice for $\mathbf{T}_\kappa(\cdot)$ to be semi-invertible at a given $\mathbf{F}$ unless the eigenvalues of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ are distinct. We now show that, *if the response function* $\mathbf{T}_\kappa(\cdot)$ *for a material of coaxial type is as in* (7), *then* $\mathbf{T}_\kappa(\cdot)$ *is semi-invertible at any deformation* $\mathbf{F}$ *at which the* B–E$^+$ *inequalities hold.*

Indeed, by a simple calculation and use of the Cayley–Hamilton theorem, one can show that (7) implies that

$$\mathbf{T}^2 = \Gamma_0\mathbf{1} + \Gamma_1\mathbf{B} + \Gamma_2\mathbf{B}^2,$$

where

$$\begin{aligned}
\Gamma_0 &= \Gamma_0(\mathbf{F}) = M_0{}^2 + III\{2M_1M_2 + IM_2{}^2\},\\
\Gamma_1 &= \Gamma_1(\mathbf{F}) = 2M_0M_1 + IIIM_2{}^2 - II\{2M_1M_2 + IM_2{}^2\},\\
\Gamma_2 &= \Gamma_2(\mathbf{F}) = M_1{}^2 + 2M_0M_2 - IIM_2{}^2 + I\{2M_1M_2 + IM_2{}^2\},
\end{aligned}$$

where $M_i = M_i(\mathbf{F})$, and where I, II, and III are, respectively, the first, second, and third principal invariants of $\mathbf{B}$. We see therefore that for arbitrary numbers H_i, $i = 1, 2, 3$,

$$\begin{aligned}
H_0\mathbf{1} + H_1\mathbf{T} + H_2\mathbf{T}^2 &= H_0\mathbf{1} + H_1\{M_0\mathbf{1} + M_1\mathbf{B} + M_2\mathbf{B}^2\} + H_2\{\Gamma_0\mathbf{1} + \Gamma_1\mathbf{B} + \Gamma_2\mathbf{B}^2\},\\
&= \{H_0 + H_1M_0 + H_2\Gamma_0\}\mathbf{1} + \{H_1M_1 + H_2\Gamma_1\}\mathbf{B} + \{H_1M_2 + H_2\Gamma_2\}\mathbf{B}^2.
\end{aligned}$$

If we compare this last with (23), we see that $\mathbf{T}_\kappa(\cdot)$ will be semi-invertible at $\mathbf{F}$ if we can find $H_i = H_i(\mathbf{F})$ satisfying the system

$$\begin{pmatrix} 1 & M_0 & \Gamma_0 \\ 0 & M_1 & \Gamma_1 \\ 0 & M_2 & \Gamma_2 \end{pmatrix}\begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{24}$$

An analysis of the system (24), using the form of the Γ_i, shows that (24) has a solution if and only if the determinant of its matrix of coefficients, $\Delta = \Delta(\mathbf{F}) \equiv M_1\Gamma_2 - M_2\Gamma_1$, is non-zero, and then, of course, that solution is unique and is easily seen to be given by

$$\begin{aligned}
H_0 &= H_0(\mathbf{F}) = \frac{M_2\Gamma_0 - M_0\Gamma_2}{\Delta},\\
H_1 &= H_1(\mathbf{F}) = \frac{\Gamma_2}{\Delta},\\
H_2 &= H_2(\mathbf{F}) = \frac{-M_2}{\Delta}.
\end{aligned}$$

Moreover, a straightforward calculation shows that the determinant Δ can be written as

$$\begin{aligned}
\Delta = \Delta(\mathbf{F}) &= M_1\Gamma_2 - M_2\Gamma_1\\
&= M_1{}^3 + 2IM_1{}^2M_2 + \{II + I^2\}M_1M_2{}^2 + \{I\,II - III\}M_2{}^3,\\
&= \{M_1 + (\beta_1 + \beta_2)M_2\}\{M_1 + (\beta_2 + \beta_3)M_2\}\{M_1 + (\beta_3 + \beta_1)M_2\},
\end{aligned}$$

where β_1, β_2, and β_3 are the three eigenvalues of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Since, for materials of the type (7), the B–E$^+$ inequalities for $\mathbf{T}_\kappa(\cdot)$ at $\mathbf{F}$ are exactly the requirement that

$$M_1(\mathbf{F}) + (\beta_i + \beta_j)M_2(\mathbf{F}) > 0 \;\; \forall \;\; i \text{ and } j,\; i \neq j.$$

we see that the B–E$^+$ inequalities at $\mathbf{F}$ suffice to ensure that $\Delta(\mathbf{F}) \neq 0$, and hence, as claimed, they are sufficient to guarantee the semi-invertibility at $\mathbf{F}$ of any response function

$\mathbf{T}_\kappa(\cdot)$ of the form (7). Trivially, the strong ellipticity at $\mathbf{F}$ of $\mathbf{T}_\kappa(\cdot)$ ensures the semi-invertibility of $\mathbf{T}_\kappa(\cdot)$ at $\mathbf{F}$.

That even strong ellipticity does *not* suffice for the *invertibility* of a general stress–strain relation may be seen by considering the isotropic elastic material for which $\mathbf{T}_\kappa(\mathbf{F}) = I^{-1}\mathbf{B}$ where $I \equiv tr\mathbf{B}$. It is straightforward to verify that this material obeys the B–E (indeed, even the B–E$^+$) and T–E$^+$ inequalities at every deformation $\mathbf{F}$. By Proposition 3, it then follows that $\mathbf{T}_\kappa(\cdot)$ is strongly elliptic in a neighborhood of every $\mathbf{F} = \alpha\mathbf{R}$, $\mathbf{R}$ orthogonal and α positive. Nevertheless, while $\mathbf{T}_\kappa(\cdot)$ is trivially semi-invertible at every deformation $\mathbf{F}$, it is impossible to express $\mathbf{B}$ as a function of $\mathbf{T}$ alone, since the entire *ray* of strains $\mathbf{B}(s) = s\mathbf{B}^\dagger$, $s > 0$ and $\mathbf{B}^\dagger$ fixed, positive definite, and symmetric, is mapped by $\mathbf{T}_\kappa(\cdot)$ onto the same fixed tensor $\mathbf{T}^\dagger = (tr\,\mathbf{B}^\dagger)^{-1}\mathbf{B}^\dagger$ of unit trace.

Finally, consider the result of Batra[5] who showed that in any isotropic material a simple tension produces a simple extension at any deformation for which the E (for empirical) inequalities† hold. This result may be extended at once to materials of coaxial type and to deformations at which merely the B–E or B–E$^+$ inequalities hold.‡ Indeed, if $\mathbf{F}$ corresponds to a state of simple tension of amount $\hat{T} \neq 0$, then $\mathbf{T}_\kappa(\mathbf{F}) = \hat{T}\mathbf{e}\otimes\mathbf{e}$ for some unit vector $\mathbf{e}$. Since $\hat{\mathbf{B}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T$ commutes with $\mathbf{T}_\kappa(\hat{\mathbf{F}})$, it now follows easily the $\mathbf{e}$ must be an eigenvector of $\hat{\mathbf{B}}$, and so, by the spectral theorem,

$$\hat{\mathbf{B}} = \hat{\beta}\mathbf{e}\otimes\mathbf{e} + \beta\mathbf{f}\otimes\mathbf{f} + \bar{\beta}\mathbf{g}\otimes\mathbf{g},$$

for an orthonormal eigenbasis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ of $\hat{\mathbf{B}}$ and a triad of corresponding eigenvalues $\{\hat{\beta}, \beta, \bar{\beta}\}$. Now suppose that the B–E or the B–E$^+$ inequalities hold at $\hat{\mathbf{F}}$. Then, since $\mathbf{f}$ and $\mathbf{g}$ are also eigenvectors of $\mathbf{T}_\kappa(\hat{\mathbf{F}})$ with common eigenvalue (equal to zero), Proposition 1 tells us that it is necessary that $\bar{\beta}$ equal β, and so

$$\hat{\mathbf{B}} = \hat{\beta}\mathbf{e}\otimes\mathbf{e} + \beta\{\mathbf{f}\otimes\mathbf{f} + \mathbf{g}\otimes\mathbf{g}\}$$

is a simple extension as claimed. Further, we still have the residual inequality that $(\hat{T} - 0)(\hat{\beta} - \beta) > 0$ if $\hat{\beta} \neq \beta$. By Proposition 1, $\hat{\beta} = \beta$ is impossible unless $\hat{T} = 0$; we thus see that $\hat{T} > 0$ (tensile loading) $\Rightarrow \hat{\beta} > \beta$, while $\hat{T} < 0$ (compressible loading) $\Rightarrow \hat{\beta} < \beta$.

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†See[1, 2] for a discussion of these inequalities. Here I only note that, in addition to implying the B–E inequalities (which is well-known), they also imply the B–E$^+$ inequalities.

‡Note added. After our manuscript was completed, Prof. Batra pointed out to us that it was observed in his note[11] that his results in[5] for isotropic materials followed if just the B–E inequalities held at F.